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Linking Prior Knowledge to New Learning: Mathematical Connections in Complex Number Instruction

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ABSTRACT

The introduction of complex numbers represents a significant conceptual transition for students, making it essential to understand how teachers connect new mathematical ideas to prior knowledge in order to promote coherent and meaningful learning. This study investigates how secondary mathematics teachers create meaningful instructional connections (Instruction-Oriented Connections, IOC) when introducing complex numbers to Grade 12 students. Drawing on Businskas' framework and subsequent research, we focus on Linking with Prior Knowledge (LinkPK) connections, links between newly introduced concepts and previously learned material that are not immediately apparent to students unless explicitly highlighted by the teacher. Six experienced teachers' online lessons were analyzed through an abductive approach, combining deductive analysis informed by the theoretical framework with inductive identification of emergent themes. Two new subcategories of LinkPK connections emerged: Extensional Connections, where new concepts are presented as natural extensions of existing knowledge, and Structural Connections, where teachers highlight how underlying properties or structures remain consistent across old and new concepts. Findings show that both types of connections play a critical role in fostering conceptual understanding by integrating new content into students' existing mathematical frameworks. Without such connections, new ideas risk being perceived as isolated facts rather than part of a coherent system. This study contributes to the Extended Theory of Connections (ETC) by refining its typology and offers pedagogical insights for advancing conceptual learning in mathematics classrooms.

Introduction

The idea of "making connections between concepts" is a recurring theme in mathematics education, particularly in two key discussions. First, it is central to the notion of mathematical "understanding," especially when distinguishing between relational and instrumental understanding (Skemp, 1976; Empson et al., 2010). Relational understanding refers to knowing both what to do and why—an integrated grasp of mathematical concepts and their interrelations. In contrast, instrumental understanding involves knowing procedures without necessarily understanding the underlying concepts. The ability to make meaningful connections between ideas is what allows learners to move from procedural fluency to deep, conceptual understanding, highlighting the relational aspect emphasized in this discourse. “The learner actively constructs knowledge and understanding through interactions between new knowledge and previous knowledge: our understanding builds and accumulates upon prior understanding” (Myhill & Brackley, 2004, p. 264). It is well established in the literature that mathematical understanding is deeply rooted in the connections learners form between concepts (e.g., Hiebert & Carpenter, 1992; Yang et al., 2021). When students are able to link new ideas with prior knowledge, they are more likely to develop a coherent and flexible understanding of mathematics. These connections help them not only make sense of abstract concepts but also apply them in varied contexts, enhancing both retention and transfer of knowledge.

Second, the ability to make connections is widely recognized as an essential "skill" in effective mathematics pedagogy (NCTM, 2000). Making connections is viewed as a skill that comprises of four components, out of which the first is making connections “between concepts” (Bingölbali & Coşkun 2016). Teachers who foster these connections help students see mathematics as a coherent and connected body of knowledge rather than a collection of isolated procedures or facts. This pedagogical stance encourages students to build on what they already know, apply concepts across different contexts, and develop more flexible problem-solving strategies. In both perspectives—understanding and pedagogy—the emphasis on connecting ideas underscores the foundational role of prior knowledge in constructing meaningful mathematical learning.

The act of making connections is recognized as a core pedagogical skill, comprising several interrelated components. One of the foundational components is the ability to establish meaningful connections between mathematical concepts themselves (Bingölbali & Coşkun, 2016). “What is meant by being sequential and cumulative is that mathematical concepts and systems are constructed on one another; they are interrelated and prior knowledge or concepts are used in defining concepts and constructing systems.” (Bingölbali & Coşkun, 2016, p. 233).

This involves recognizing how different ideas relate, overlap, or build upon each other-such as the connection between proportional reasoning and linear functions, or between area models and algebraic expressions. Teachers who are skilled in making and highlighting such connections help students see mathematics as an interconnected system rather than a set of disconnected topics. Furthermore, the process of learning a new concept is most effective when it is intentionally linked to what students already know (Bransford et al., 2000). This principle aligns with the broader understanding of learning as a constructive process. Teachers who are proficient in this aspect of pedagogical content knowledge are better able to scaffold new ideas, build upon familiar contexts, and facilitate deeper learning. As Tchoshanov (2011) notes, such pedagogical proficiency is positively associated with improved student achievement, reinforcing the critical role that conceptual connections play not just in understanding, but also in successful teaching and learning. Successful teaching of mathematics requires justifying mathematical claims through logically deduced connections among mathematical ideas (Heid et al., 2015).

This study adopts Businskas' (2008) definition of a "mathematical connection" as "a true relationship between two mathematical ideas A and B" (p. 18). The introduction of the model by Businskas (2008) which describes fine-grained mathematical connections contained in quadratic functions and equations, has led to a strand of research investigating mathematical connections.

Classification of Mathematical Connections

In mathematics, the importance of at least three kinds of connections is identified as particularly beneficial: connections within mathematics, across the curriculum, and with real-world contexts (Monroe & Mikovch, 1994). The terms intra-mathematical and extra-mathematical are also used in the literature to refer to connections within mathematics (between concepts, theorems, procedures etc.) and connections outside of mathematics (connections with different disciplines and real life). There are many studies that have looked into real-world connections made by teachers (e.g.; Gainsburg, 2009; Özgeldi & Osmanoğlu, 2017; Rathnayake & Jayakody 2023) and by students (Altay et. al., 2017), which fall into extra-mathematical connections.

The most frequently used model for mathematical connections is the one proposed by Businskas (2008). In this model, a mathematical connection is 'a true relationship between two mathematical ideas, A and B' (p.18); and five categories are identified: different representations, implication, part-whole, procedural, and instruction-oriented connections.

Other studies that used this model have built and expanded on it, adding more types of connections. The participants and the context of these studies vary. While some studies have looked at how students make connections (eg.; García-García & Dolores-Flores 2017), others have investigated connections made by preservice teachers (e.g.; Evitts, 2004; Şahal et al., 2024) and in-service teachers (e.g.; Rodríguez-Nieto, 2022; Hatisaru, 2023, Mhlolo et al, 2012). Out of the studies conducted with teachers, the connections have been investigated during teaching (e.g.; Bingölbali & Coşkun 2016, Mhlolo et al, 2012), and during teachers are engaged in problem solving (Evitts, 2004) or a mathematical task (e.g.; Eli et al, 2011; del Carmen et. al., 2024). Among studies that looked at connections made during teaching, the lessons have been on various topics. For instance, integers (De Gamboa et. al., 2023), derivatives (Rodríguez-Nieto et al, 2022), functions (Businkas, 2008), algebra (Fyfe et. al., 2017), linear programming (Jingga, 2019), geometry (Eli et al, 2013).

Our Focus: Instruction-Oriented- Linking with Prior-Knowledge

The focus of our study was in intra-mathematical connections, specifically between mathematical concepts and procedures, that are made by secondary teachers during teaching the lesson of Complex Numbers, specifically that draws on prior knowledge. That is, by making some form of connection between a newly learnt concept or procedure and a previously learnt concept or procedure. This type of connection is what is named as Instruction-Oriented Connections (IOC) in the literature on mathematical connections. When this connection was first introduced by Businkas' (2008) in her study, she identified 3 kinds of IOCs; curriculum, vocabulary and prerequisite knowledge in the context of teachers conversing about quadratics. This type can be broadly described as connections with previous knowledge.

An instruction-oriented connection can be manifested both when a new concept is linked to prior knowledge, and when an understanding of a new concept is dependent on the understanding of other concepts or prerequisites (Hatisaru, 2022). According to Hatisaru (2022), they appear when new knowledge is built on the existing knowledge, or particular mathematical concepts/skills must be known in order to understand new concept/ skill, or when previously taught/learnt knowledge is revisited.

A sub-category of IOC was identified in a study investigating mathematical connections established in teaching of functions in Hatisaru's study (2022) which is revisiting taught knowledge. These connections occur when students are reminded about what they have been taught or learnt previously. We use this as a conceptual framework for our study.

In a study conducted by Rodríguez-Nieto and collegues (2022), IOC is described as below:

> *It refers to the understanding of a concept C based on two or more previous concepts A and B, required to be understood by a subject. This type of connection is manifested in two ways: (a) association of a new topic with previous knowledge, (b) mathematical concepts and procedures connected to each other are considered prerequisites, abilities or vocabulary that students should control before the development of a new concept.* (pg. 1235)

Hatisaru (2024) captures IOC as referring to "incorporating the relations among mathematical ideas in the teaching of mathematics. They can be manifested both when a new concept is linked to prior knowledge, and when an understanding of a new concept is dependent on the understanding of other concepts or prerequisites. An additional connection (revisiting taught knowledge) occurs when students are reminded about what they have been taught or learnt previously." (pg. 586)

In a latest paper that reflects on the Extended theory of Connections (ETC), Rodríguez-Nieto (2025) summarizes IOC as "the understanding and application of a mathematical concept D derived from two or more related concepts, B and C" (pg. 117). He describes 2 forms of IOC:

1) the relationship of a new topic with previous knowledge, and

2) the mathematical concepts, representations, and procedures connected are considered fundamental prerequisites that people must have to develop new content

Therefore, it seems that IOC can be mainly of two types as identified in these studies. For the easy reference in this paper we summarize and denote them as follows with examples:

- Linking to prior knowledge (LinkPk)

Example 1: Showing how a triangle is half of a rectangle and so its area would be half of the corresponding rectangle (instead of just presenting the area formula of a triangle)

- Revisiting as pre-requisite knowledge (RevisitPK)

Example 1: Reminding the operation factorial, in order to teach Binomial Theorem

Example 2: Reminding the students what a function is and how to find the value of a function at a given point, in order to teach limit of a function

LinkPk refers to a form of instructional connection that is not immediately apparent to learners unless it is explicitly highlighted by the teacher. The type of IOC examined in this study is LinkPk, which aims to strengthen students' conceptual understanding by making deliberate links between newly introduced ideas and previously learned concepts or procedures. Such connections are typically not discovered spontaneously by learners but must be purposefully constructed and emphasized by the teacher. We argue that this form of IOC is particularly valuable for fostering conceptual understanding, as it enables learners to recognize relationships between concepts and to view mathematics as a coherent and integrated discipline, rather than as a collection of fragmented and disconnected topics.

Our objective is to identify instances in which upper-secondary teachers establish connections that foster conceptual understanding, as opposed to purely procedural understanding, when introducing complex numbers. Specifically, we seek to examine how teachers build on students' prior knowledge by creating purposeful links that fall within the category of LinkPk. Accordingly, our research is guided by the following question:

What forms of mathematical connections, grounded in prior knowledge and categorized as LinkPk, can teachers establish during the introduction of complex numbers?

Research Methods

This study focused on the topic of complex numbers, which is introduced at Grade 12 and represents students' first formal encounter with the concept. The topic was selected because it offers rich opportunities for teachers to build on students' prior mathematical knowledge and establish meaningful connections between existing and new concepts. The dataset consisted of approximately 14 hours of video-recorded lessons collected from six Advanced Level mathematics teachers who were purposively selected based on their reputations as successful and experienced educators. This deliberate selection was necessary because the types of instructional connections investigated in this study depend heavily on teachers' mathematical content knowledge and pedagogical content knowledge. All lessons were conducted online in the post-COVID educational context. Among the participating teachers, three held either a BSc in Engineering or a BSc Special Degree in Mathematics, while four possessed postgraduate qualifications in mathematics or mathematics education. The lesson recordings were transcribed verbatim and served as the primary source of data for analysis. Data analysis followed an abductive approach, combining deductive analysis informed by the theoretical framework with inductive identification of emergent themes. Initially, the two components, RevisitPK and LinkPK, of the a priori ETC category IOC were used as an analytical framework to guide the coding process. At the same time, the analysis remained open to the identification, refinement, and extension of different forms of LinkPK that emerged from the data. The analysis focused specifically on instructional episodes that established explicit connections with students' prior knowledge. Instances in which teachers merely referred to a previously learned concept, procedure, algorithm, or idea for use in the current context were categorized as RevisitPK. In contrast, instances that extended beyond simple recall and created meaningful conceptual links between previously learned and newly introduced mathematical ideas were categorized as LinkPK. Through this iterative process of moving between theory and data, the study sought to identify and characterize the different ways teachers connected prior knowledge when introducing the concept of complex numbers.

Findings

Our analysis identified instances of IOC, specifically LinkPk as guided by the research question. Two types of LinkPk connections emerged, which we name as Extensional and Structural.

- Extensional Connections are when a teacher explicitly links a newly introduced concept as an extension of a previously learned concept under new conditions.
- Structural Connections are when a teacher explicitly relates the structural properties of a previously learned concept or procedure to a newly introduced concept or procedure.

Next, we present a series of illustrative episodes representing each type of connection. The format of these examples is not uniform. Instances where a particular type of connection can be captured within a brief segment of instruction are presented in tabular form. In contrast, examples that unfold over a longer duration are described narratively, supplemented with visuals from the lessons to maintain conciseness. In summary, the mode of presentation varies according to the nature and extent of the connection as enacted during instruction. Further, when presenting lines from the transcripts, they are labeled starting from 1 in each table for clarity.

Extensional Connections

What we have come to name as "Extensional connections" occur when a teacher helps students see a new concept as a natural extension of something they already know, but now applied under different or broader conditions. For example, after students learn about whole numbers, the teacher might introduce negative numbers as an extension of the number line they are familiar with, showing how the same ideas expand to include values below zero. Similarly, exponents as an extension of multiplication, where repeated multiplication grows naturally out of the idea of repeated addition. Such connections help students build on existing knowledge rather than viewing each new topic in isolation.

When we first identified extensional type connections in the teaching of complex numbers of these 8 teachers, some of them were not as elaborate as we would have liked them to be. But we could see the potential in certain utterances that could lead to a full-blown extensional type. Therefore, we have counted these instances as well as belonging to extensional connections.

In what follows, we present 7 examples of extensional connections during teaching of complex numbers.

Extensional connections Case 1: Extending how previously learnt different types of numbers were created to the way complex numbers were created

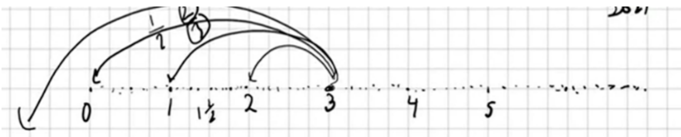
Teacher E began the lesson in a storytelling style, describing how different types of numbers emerged over time to meet human needs and solve particular problems. He revisited the number systems the students had previously encountered, explaining the historical and conceptual reasons for the introduction of each. Building on this narrative, he gradually led the discussion to complex numbers, drawing explicit connections between the evolution of earlier number systems and the need for the development of complex numbers.

In what follows, excerpts from the classroom transcript are analyzed as an instance of an Extensional Connection, where the teacher extended students' prior knowledge of how different number systems were created to the introduction of complex numbers. The Table 1 shows how the teacher linked students' prior knowledge of how different types of numbers were historically created, either to address real-world needs or to extend the number system to accommodate solutions of new types of equations, to the introduction of complex numbers.

> *"So all the types of numbers you know so far were created by humans for a reason, they were created to meet a requirement, sometimes in life, sometime in mathematics itself."*

Table 1 How different types of numbers were historically created (Teacher E)

Type of number	Narration	Reason for creation
Counting	….During those times there were no numbers. And when they took their herds of sheep out in the morning and brought back in the evening, they had no way of checking whether the same number of sheep were there. …..So counting numbers were created so that they had a way of saying how many in a group.	To address a real-world need
Fractions	….When they wanted to talk about part of a whole, they had a requirement to come up with a number to describe that. So they created fractions with its notation. You can see how a fraction uses two numbers but how it represents a single number.	To address a real-world need

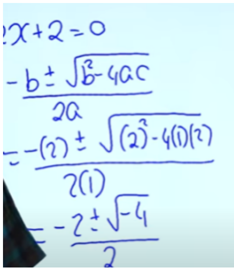
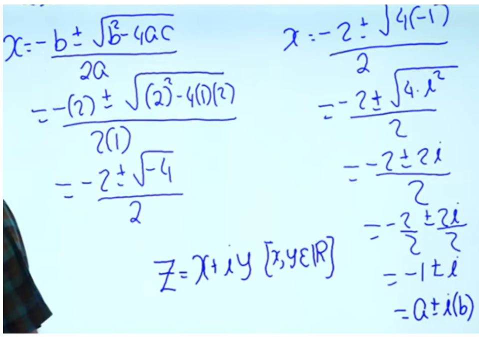
Negative numbers	….Algebra was getting developed at this time and mathematicians were solving simple equations…When they were solving equations such as, for example, like this, x plus 5 is equal to 3. Suppose you come to this step like, you are solving an equation and you finally come to this step and in the next step you would do, or you would have to do x is equal to 3 take away 5 and people used to say,.. there was a time that when people said there is no solution here. Why? What is.. look at your number system. It starts from zero. One, two, three, three take away 5, you can't take away 5 from 3 so there's no solution. Is this familiar? Saying no solution? Do we not say no solution for certain equations. Now we have an answer for this, but there was a time when people said no solution for that because they did not have a number to represent 3 take away 5 until.. they started creating. $x+5=3$ $x=\underline{3-5}$ no solution $=$ So 3 is here. If I take away 1 from 3, go back 1, so I get 2, and if I take away 2 from 3, I go back here, I end up at 1…3 take away 3, that is ok, I can do that, 3 take away 3, I go back here, 3 take away 4.. where to go? You fall off of the …so there's nothing to…  But then somebody was brave enough to say hey and …. The thing was it didn't violate the rest of mathematics … I mean, it was rather creating a more complete picture of it. Why no, hey, lets say what could be before 0, so this whole negative concept, the opposite of these numbers, what	
Complex numbers	..and then again when people were solving equations we came to ,we stumbled upon equations like this say x squared plus 1 equals 0 and then say you're solving an equation and you come to this step so you would go x squared equals minus 1 and then you square root it then square root of minus 1… oh oh no no solution this is absurd there is no solution, $x^2+1=0$ $x^2=-1$ $x=\sqrt{-1}$ no sol. there is no square root for negative one, why did we say that because we don't have that number on this line..if you want to think of square root of negative 1	To accommodate solutions of new types of equations

The narrative intentionally builds on students' prior knowledge of the different types of real numbers and the historical motivations underlying their development, extending this line of reasoning to the creation of complex numbers. In doing so, complex numbers are situated within the broader number system, allowing them to be meaningfully assimilated as a natural continuation of human efforts to address mathematical needs—just as with the earlier extensions that gave rise to other classes of numbers.

Extensional connections Case 2: Extending Real solutions of a quadratic equation to non-Real solutions.

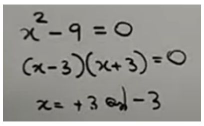
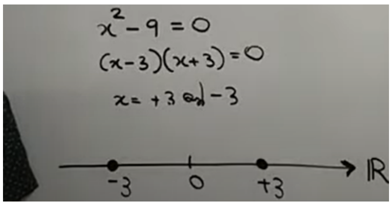
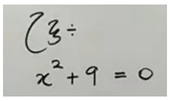
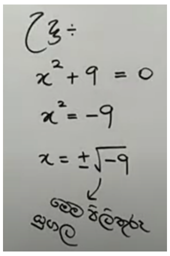
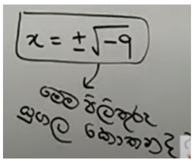
In the following approach taken by Teacher A as shown in Table 2, he takes a quadratic equation that has no real factors. i.e. whose solutions are not Real. He uses the quadratic formula in order to encounter a square root of a negative number in the discriminant.

Table 2 Extending from solutions that are not real to introducing complex (Teacher A)

	Utterance	Accompanied writing	Deciding the IOC type	IOC type
A1	We have learnt the lesson of solving quadratic equations	writes the equation $x^2 + 2x + 2 = 0$ on the board	“solving quadratic equations” is a previously learnt lesson and it is reminded.	RevisitPK
A2	We can solve this by using completing square or using the quadratic formula	writes the quadratic formula and substitutes values from the equation to it and then starts simplifying	Method of solving QE by using quadratic formula is reminded.	RevisitPK
A3	Here we see that, what comes inside this bracket is square-root of -4. We know the square-root of 4. But with the negative sign, we say this square-root of -4 is not Real.We know the square-root of 4. But with the negative sign, we say this square-root of -4 is not Real.		We know...but	LinkPk (extension)
A4	We put something called i^2	simplify √(-4) as √(4(-1)).		
A5	the square root of -1, we call as i			
A6	Therefore, we define a complex number like this. If we take the complex number to be z, we say x+iy			

The Table 3 below shows how teacher C enacted this during his lesson. He starts the lesson saying that I'm taking a different approach to the lesson this time". He starts with a quadratic equation whose solutions are Real and then takes a second equation whose solutions "cannot be marked on the Real line".

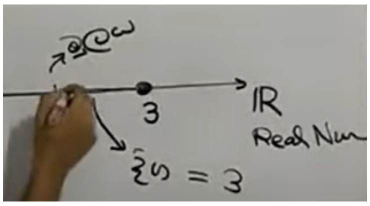
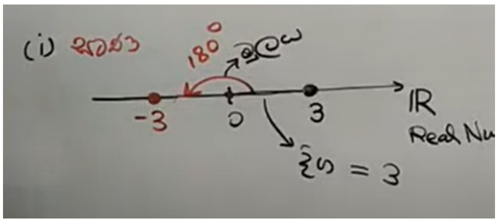
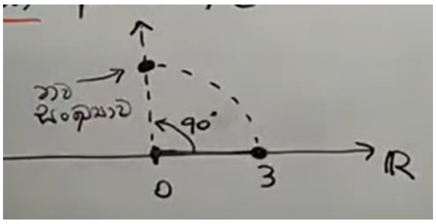
Table 3 Where to mark the new solutions on the number line (Teacher C)

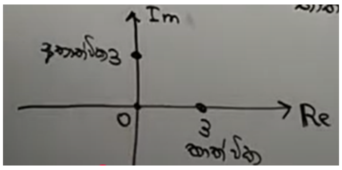
	Utterance	Accompanied writing	Deciding the IOC type	IOC type
A1	If a number can be marked on the Real number line, then it is a Real number.			
A2	If you have to solve an equation of this form.., this becomes $x - 3$ times $x + 3$, and we get $x = 3$ and $x = -3$			RevisitPK
A3	These two answers, kids, we can mark on the Real number line. 0 is here. 3 is here and -3 is here. So the values of x that makes this equation equal to 0, we could mark on the Real number line.			RevisitPK
A4	Next example, look.. Now we have to find some two values for x, that would satisfy this equation.			
A5	We will use our common sense and solve this. Here when we take the square-root of both sides, we get ±√9 Now here, where do we mark this pair of answers?			
A6	So far, all the solutions we got for our equations, we could mark using the Real axis. On it. But where to mark these two answers? Where are they?	 Where is this pair of solutions? (translations of the text on the board)	Connecting previous solutions that could be 'marked' on the Real line to solutions that cannot be marked on it.	LinkPk (extension)

Extensional connections Case 3: Extending the idea of deriving known real numbers by rotating another real number, to the derivation of imaginary numbers.

Teacher C starts a discussion under the title "Deducing various numbers from a Real number" (See Table 4). He starts by showing how a negative number can be rotated by 180^0 to obtain a positive number and then extends this idea to rotate a positive number by 90^0 to obtain a ‘new type of numbers’. The following excerpt illustrates this.

Table 4 Deducing various numbers from a Real number (Teacher C)

	Utterance	Accompanied writing	Deciding the IOC type	IOC type
C1	In the Real number system, 0 is here. 3 is here. What’s the distance to 3 from the origin? 3 okay?			
C2	Now I rotate this by 180^0 in an anti-clockwise direction. Who do I find? This must be the same distance. If we rotate a Real number by 180^0 who do we find? Either positive 180^0 or negative 180^0? We find a negative number.		Building a new idea with known numbers	
C3	So you see, we deduced a negative number from a real number. From 3, we created -3 by rotating..		Building a new idea with known numbers	RevisitPK
C4	This means we can deduce a negative number by rotating a positive number or can deduce positive numbers by rotating negative numbers.		Building a new idea with known numbers	
C5	Using this idea, we can also find this (pointing to ±√(-9))		Connecting the same idea to a new type of number	LinkPk (extension)
C6	Now watch carefully. Relative to 0, I rotate the line segment 3, by 90^0. Then kids, who do I find? I find a new number who’s right here. This new number that I can mark here….not only by 90^0, you can also rotate by 30^0, 15^0, 175^0, but my target is to find those, so I have rotated by 90^0. Here we find a new number.			

C7	Such numbers whose perpendicular to the Real axis are called imaginary axis. So, this is real 3 and this is imaginary 3.		Creating new knowledge

Through introducing the idea of deriving a known type of number (negative numbers) from a ‘real number’ (he must have meant positive real) by rotating it, this teacher extended the same idea to creating √(-9).

Extensional connections Case 4: Extending the meaning of multiplication as scaling and rotating

Teacher E talks about extending the meaning of multiplication from what students know ‘a certain number of groups of another number’ to re-conceptualizing multiplication as a ‘scaled rotation’. He starts the discussion saying that “Let's revisit multiplication. we are going to think about multiplication and give it a new meaning.”.

He explains that the multiplication between any two numbers can start with 1, which is represented by a “directed line segment” (See Figure 1).

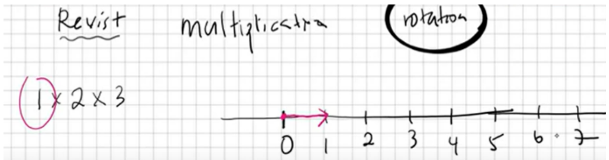


Figure 1 Explaining multiplication on the number line (Teacher E)

He goes on to explain 2×3 as 1×2×3 and how multiplication by a positive number is related to a rotation of 360^0 and the multiplication by a negative number is related to a rotation of 180^0 while scaling the line segment by the magnitude, as shown in Figure 2.

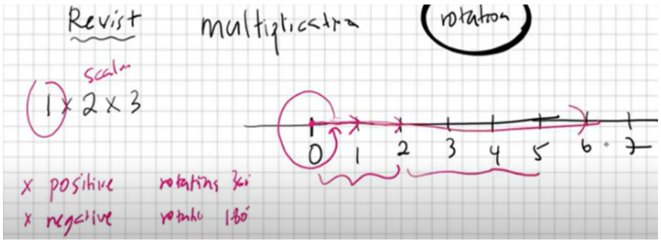


Figure 2 Multiplication as rotation (Teacher E)

He goes on to explain, with the drawing on Figure 3, the multiplication of two negative numbers to show how ‘minus into minuts is plus’ rule makes sense from this new conceptualization.

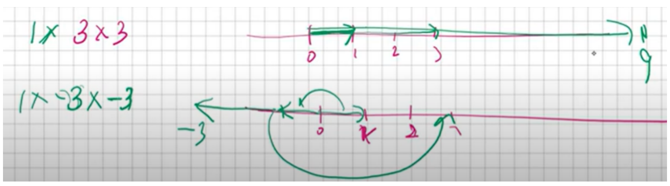


Figure 3 New conceptualization of multiplication (Teacher E)

And then the question is posed to the students, with the drawing on Figure 4, where this number √(-1) should be if we are to start with 1 and need to end up on -1 after performing the product 1×√(-1)×√(-1).

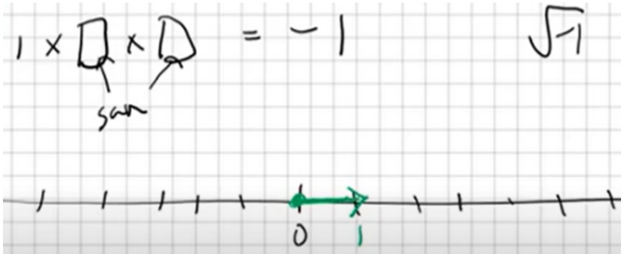


Figure 4 Where should √(-1) be? (Teacher E)

Students say that this number should have the same size as 1 since the length does not change at the end and since its two equal rotations to end up on -1, each one should be a rotation of 90^0.

Extensional connections Case 5: Extending 1-dimentional Real number line to 2-dimentional complex number plane

Teacher E introduced complex numbers through a visual approach to the number system, transitioning from the familiar concept of the real number line to a two-dimensional plane of numbers. In this framing, complex numbers emerged as a natural extension of the existing system, moving from one dimension to two, rather than being presented merely as entities of the form $a + bi$.

This visual entry point also enabled the construction of a broader narrative of number development: beginning with discrete dots representing the counting numbers, extending to a seemingly continuous line through the inclusion of fractions, broadening further with negative numbers, and subsequently filling the gaps with irrational numbers, until the real line appeared complete. At this stage, the extension to a plane was justified by the absence of space within the one-dimensional line to accommodate a new type of number. The following excerpt in Table 5 illustrates how Teacher E enacted this extensional connection after reconceptualizing multiplication as scaled rotation. It is a student who suggests that we 'come out of the line'. The following exchanges via chat and voice followed after Techer E's prompt, where should this number be?

"Is there a way that I can rotate on this plane so that… you want to create this number. You want to find a place for it in this picture. I want to create a spot for him. Where can I place him?" (See Figure 5)

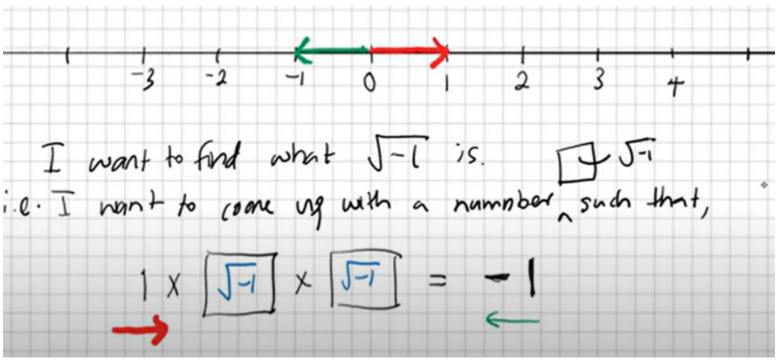


Figure 5 Where can √(-1) be placed on the number line? (Teacher E)

Table 5 Chat massages exchanged (Teacher E)

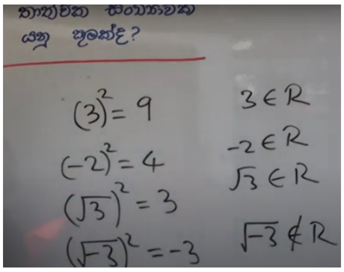
	Chat messages and voice
Student 3	90 left and 90 left, does that make sense?
Student 1	Ms. That was what I was trying to say, so, if, like if we want to get -1, should we like go out of the line
Teacher	Of course!! Yes! Yes!
Student 1	Then drop back to the line?
Teacher	This is so difficult to do because we are stuck on this line. We are stuck in one dimension.
Student 1	Now Math sounds different to me
Student 2	I feel like I'm seeing math with a new lens. Who else is mind blown?
Teacher	Yes! Now see how hard it is for us to do? Now we think numbers are on this line na. We are bound on this line. You have to break free. You have to come to two dimention
Student 1	Ms. I'm guessing that 90 left is, coming out of the line to that imaginary point and then falling back
Teacher	Exactly! So with this new way of thinking about multiplication, I'm extending the story.
Teacher	That means your Real number line where all your Real numbers live.. what just happened is this… Square roots of all the negative numbers live on this line. We just extended. Numbers don't just live on a line, they live on a plane. There is a whole plane of numbers.

Here the teacher is extending numbers 'living on a line', to numbers 'living on a plane'. And student fascination is apparent from their messages in the chat box.

Extensional connections Case 6: Extending square-roots of positive numbers to square-roots of negative numbers

Teacher B started lesson with defining what a real number is. He puts the title on the board, "what is a Real number?". He then asks "In what way can we identify a Real number?", and answers, "A Real number when squared is definitely a non-negative number." (See Table 6)

Table 6 From square-roots of positive number of square-roots of negative numbers (Teacher B)

	Utterance	Accompanied writing	Deciding the IOC type	IOC type
B1	Is 3 Real? Yes, because when squared, you get 9, which is non-negative.		Goes from known Real numbers $3, -2, \sqrt{3}$ to a non-Real number $\sqrt{-3}$ through a common definition for all Real numbers.	LinkPk (extension)
B2	Is -2 Real? Yes, because when squared, you get 4.			
B3	Is $\sqrt{3}$ Real? Yes, because if you square it, you get 3, which is non-negative.			
B4	Tell me, is $\sqrt{-3}$ Real? If you square square root of -3, you get -3, it is negative. Therefore $\sqrt{-3}$ is not a Real number.			

The teacher then discusses how the value of √(-3) could be found (See Figure 6). He introduces i as a number that a mathematician has defined such that $i^2 = -1$ and then writes √(-3) using i.

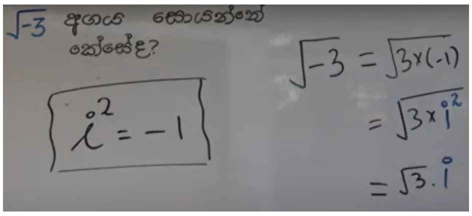


Figure 6 How the value of √(-3) could be found (Teacher B)

Extensional connections Case 7: Extending Real number set to Complex number set

After introducing complex numbers as numbers that can be written in the form a+bi, Teacher E contextualized the concept by revisiting the classification of numbers as sets, representing them within a Venn diagram (See Figure 7). In doing so, he extended the number system familiar to students into a broader framework encompassing all previously encountered number types. He illustrated this using a slide containing the following Venn diagram.

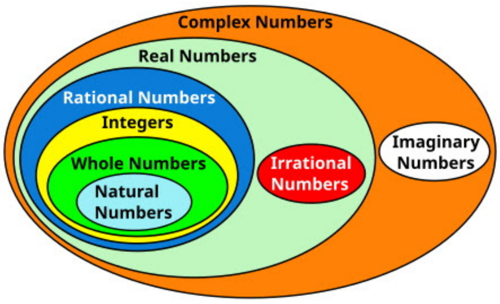


Figure 7 Venn Diagram of numbers (Teacher E)

This episode illustrates an extensional connection because the teacher expanded the students' existing understanding of number sets to incorporate complex numbers, thereby extending the familiar structure of the number system into a more comprehensive system.

The Table 7 below gives a summary of the different instances of extensional type connections made by teachers.

Table 7 From square-roots of positive number of square-roots of negative numbers (Teacher B)

Prior knowledge	Extension	New knowledge
Previously learnt classes of numbers within Real numbers	The need that arose to create them	How a similar need gave rise to complex numbers
Finding Real solutions of quadratic equations	When solutions of a quadratic equation are non-Real	Introduction of i to write non-real solutions of quadratic equations
Real numbers	How negative numbers can be derived from positive numbers	Introduction of i by extending the same idea
Meaning of multiplication ($a \times b$ as a groups of b)	Multiplication as rotation and scaling	Placement of imaginary numbers on a line perpendicular to the Real line
1-dimentional Real number line	Accommodate 'non-real roots' of quadratic equations	Extending to the 2-dimentional Complex plane
Square-roots of positive numbers	Square-roots of negative numbers	Introduction of i
Largest number set is the set of Real numbers	Real number set is a sub-set of complex number set	Largest number set is the set of complex numbers

Structural Connections

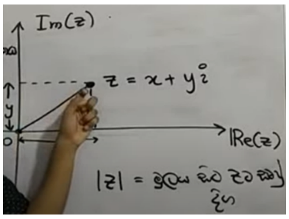
Structural connections occur when a teacher highlights how the underlying structure or properties of a familiar concept also apply to a new one. For example, after students learn the distributive property in arithmetic, the teacher might show how the same property holds in algebra when expanding a(b+c) or later in calculus when working with integrals. Similarly, the symmetry and properties of operations on real numbers can be connected to the same structures when students begin working with complex numbers.

In what follows, we present 3 examples of structural connections made during the teaching of complex numbers that emerged from data.

Structural connections Case 1: Structural similarity between the magnitude of a real number and the magnitude of a complex number

Under the topic titled Properties of $z = x + iy$, as written on the board, Teacher C introduces the modulus of a complex number. Following the explanation of the conjugate, Teacher C transitions to the modulus as the second key property (See Table 8).

Table 8 Modulus of a complex number (Teacher C)

	Utterance	Accompanied writing	Deciding the IOC type	IOC type
C1	So we have learnt the modulus of a Real number. It is, the distance to a Real number from 0.		Recalling previous knowledge	RevisitPK
C2	Now, these new numbers also have a size			LinkPk (structure)
C3	And that too is how far that number is from 0.			
C4	So if you have a complex number z, the modulus of z also means the distance from 0 to z.			

Teacher C's narration illustrates how he connects students' prior knowledge of the modulus of a real number, defined as the distance of a real number from zero, to the concept of the modulus of a complex number. Rather than presenting it as a completely new idea, he emphasizes the continuity of meaning: the modulus of a complex number similarly represents its distance from zero. This reflects a structural connection, as the new concept is explicitly linked to an existing mathematical idea through a shared underlying structure.

Structural connections Case 2: Structural similarity between Pythagoras' Theorem and the formula of the magnitude of a complex number

Teacher E also explained the modulus of a complex number by recalling first the modulus of a Real number as the distance from 0, similar to the way Teacher C did, as shown in Figure 8. Then she marks a complex number $4 + 3i$ on an Argand diagram and draws this distance from 0 to the number.

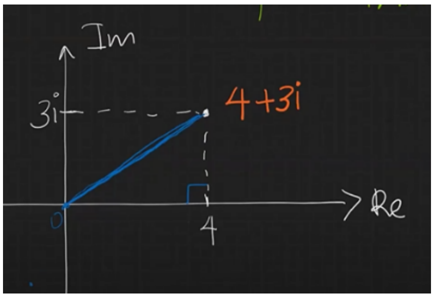
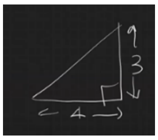
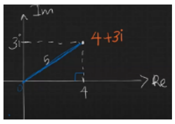


Figure 8 Modulus of a complex number (Teacher E)

Then the teacher posed the question, "what is the length of this line?"

Three students say '5' in the chat box. The utterances E1 to E5 in Table9 illustrate a structural type of LinkPk connection.

Table 9 Structural connection enacted (Teacher E)

	Utterance	Accompanied writing
E1	Very nice! How did you find that? Yes! Pythagoras theorem. There is a right angled triangle.	
E2	Let me draw that triangle separately on the side here… so that you can see it clearly..	
E3	This side is 3. This side is 4.	
E4	Then this is square root of 4^2 plus 3^2, which is 5.	
E5	Therefore, this distance is 5…that means the modulus of this number is 5.	

It is only after explaining how to find that distance using Pythagoras' theorem, did the teacher E establish the formula for the modulus of a complex number as $|z| = \sqrt{(a^2 + b^2)}$ as shown in Figure 9.

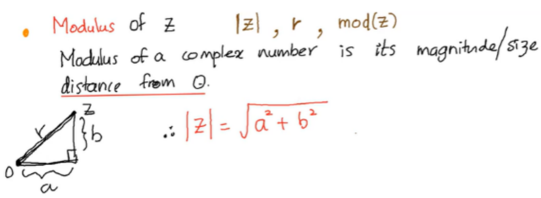


Figure 9 Modulus and Pythagoras's theorem (Teacher E)

Structural connections Case 3: Structural similarity between dividing a number by a decimal or a surd and dividing by a complex number

When arriving at the division of complex numbers under the topic of 'operations with complex numbers', Teacher E took the division of $2+3i$ by $1-2i$ as an example as shown in Figure 10.

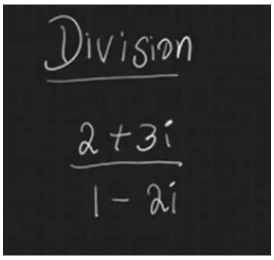


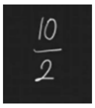
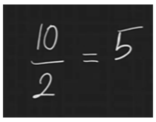
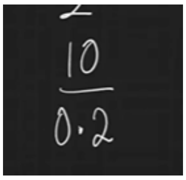
Figure `10 Structural connection (Teacher E)

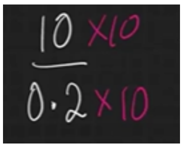
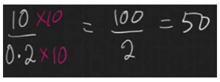
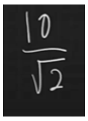
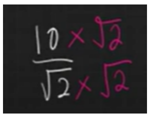
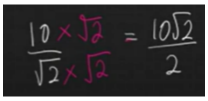
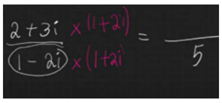
He then posed the question of how to approach the problem:

"How do we tackle this? With addition, subtraction, and multiplication, you were able to apply the same methods you use with algebraic expressions. But what about this one? How should we proceed?"

The next few minutes of the lesson exemplified a structural connection. The teacher invited students to recall how division operates across different types of numbers. His strategy was to fix one number and then divide it by various categories of numbers, prompting students to recognize structural similarities and differences across number systems (See Table 10).

Table 10 Structural connection across division (Teacher E)

	Utterance	Accompanied writing	Deciding the IOC type	IOC type
E1	Think about 10 divided by 2. Let's go back to the basic idea of what division means.		Activating PK on the meaning of division.	RevisitPK
E2	10 and 2 are also complex numbers but they are pure Real and just whole numbers.			
E3	So when we have $\frac{10}{2}$, how do we that?			
E4	There we think how many 2s go into 10. That's one way to think about division. So it's 5.		Division of whole number by a whole number.	RevisitPK
E5	What about…let's look at division of different types of numbers. How do we divide, for example, if I have 10 divide by..say..a fration..if I write it in decimal representation..0.2		Division of whole number by a whole number.	
E6	How do we tackle one like that? Still we can think of it like how many 0.2s go into 10 or…0.2 times		Reminding other meanings of division	RevisitPK

	what gives me 10. There are different ways to think about it.			
E7	But as a strategy there is something that we do here. Whenever we are dividing by a fraction written in decimal form…something that we do is, if you remember, to make it easy..to make the division easy, you can multiply both the top and the bottom by 10.		Recalling a strategy for division by a decimal number.	RevisitPK
E8	The reason why we do this is, to make the divided into a whole number.			
E9	So the top becomes 100 and the bottom also becomes a whole number now..2,..and now its easier to think and we know that the answer is 50.			
E10	And lets take another one. Division by another kind of number. Lets take 10 divided by $\sqrt{2}$		Recalling a strategy for division by a surd.	RevisitPK
E11	Again we use this same strategy. Both numbers we multiply by some number so that the bottom number is easier to work with. Here $\sqrt{2}$ is the problematic number, the divisor. So what we do here is, to make it rational, we multiply both by $\sqrt{2}$. We call it rationalizing the denominator.			
E12	And now you get $\sqrt{2}$ divided by . Now its easy to do. The answer is $\sqrt{2}$			
E13	So we use a similar strategy here. The problem here is having to divide by a complex number. If I had $\frac{2+3i}{2}$, then it can be done easily. So, we think, what can I multiply this denominator by, so that it becomes a pure Real number. Can someone suggest?		Connecting this previously done strategies and showing that it is a similar thing we do here.	LinkPk (Structural)
E14	You multiply both the top and bottom by the conjugate of $1-2i$. This makes the divisor a Real number and then we can do the division easily. You can even call it 'realizing' the denominator!			

The above three examples illustrate how teachers establish structural connections between newly introduced concepts and previously learned material. Specifically, the modulus of a complex number retains the same meaning as the modulus of a real number; its formula reflects the structure of the Pythagorean theorem, as it directly applies this result; and the strategy of multiplying by the conjugate when dividing by complex numbers parallels techniques students have previously encountered.

Table 11 summarizes the structural connections identified in the study.

Table 11 Examples of Structural Connections

Prior knowledge	Structure	New knowledge
The magnitude of a real number	The distance from zero to that number	Magnitude of a complex number
Pythagorean Theorem	$c = \sqrt{a^2 + b^2}$	Magnitude of a complex number $\sqrt{a^2 + b^2}$
Division by a decimal number ($\frac{n}{0.1} \times \frac{10}{10}$)	Making the denominator an integer	Division by a complex number (making the denominator real) $\frac{n}{ai+b} \times \frac{ai-b}{ai-b}$
Division by a surd ($\frac{n}{\sqrt{m}} \times \frac{\sqrt{m}}{\sqrt{m}}$)	Making the denominator rational	

It is noteworthy that we could not observe any LinkPK type of connections from two teachers in our sample. This could be hinting to the fact that, as we envisioned, LinkPK type connections are part of subject specific pedagogical knowledge, named as specialized content knowledge (SCK) in the Mathematical Knowledge for Teaching framework (Ball & Bass, 2003).

Discussion and Conclusion

There is a profound relationship between mathematical connections and mathematical understanding (García-García, 2024; Cai, J. & Ding, 2017). In this study on mathematical connections, our focus was on Instruction-Oriented Connections (IOC), that is, the connections teachers intentionally establish during instruction. Our attention on IOC stemmed from our conviction that for educators and researchers, leveraging these connections is key to advancing mathematical learning and effective pedagogy that encourages conceptual understanding of mathematics. Drawing on the literature, we conceptualized two types of IOC: RevisitPK and LinkPK. The latter refers to connections that are not immediately evident to students unless explicitly highlighted by the teacher. Our interest centered on LinkPK, as such connections hold particular potential for fostering conceptual understanding. Accordingly, our guiding question was: What forms of mathematical connections, grounded in students' prior knowledge and categorized as LinkPK, can teachers establish to meaningfully support students' understanding? The instructional context for this investigation was the teaching of complex numbers.

García-García and Dolores-Flores (2018) highlight the need for further research to validate existing typologies and identify additional categories not yet included in the ETC framework. Our study contributes to this discourse by validating two previously established types of connections RevisitPK and LinkPk while introducing two new types under LinkPk: Extensional Connections and Structural Connections. We argue that both types play a crucial role in fostering conceptual understanding, particularly in the teaching of complex numbers.

Extensional connections occur when teachers help students view a new concept as a natural continuation of existing knowledge, but under broader or different conditions. In contrast, structural connections arise when teachers emphasize that the underlying meaning, structure, or properties of a familiar concept remain applicable to a new one. Both types involve linking prior knowledge to new material in ways that go beyond simply recalling prerequisite information. Rather, they represent deliberate pedagogical choices made by teachers who recognize the value of such connections in fostering conceptual understanding, an understanding characterized by the interrelatedness of mathematical ideas (Skemp, 1976).

A defining feature of both types of connections identified in this study is that newly introduced concepts and procedures were purposefully linked to students' existing knowledge. Such connections, whether extensional or structural, facilitate the construction of meaningful relationships within the content, thereby supporting deeper conceptual understanding. We note that all the concepts and procedures illustrated in the study could, in principle, be taught without making such connections. However, in the absence of these links, the newly learned concept or procedure would remain a stand-alone, isolated piece of knowledge, disconnected from the broader mathematical framework. Consequently, the critical criterion for identifying a LinkPk lies in determining whether the new concept or procedure transcends mere reliance on prior knowledge to actively build upon and integrate with it in a substantive and coherent manner.

At the core of our study, situated within the broader framework of ETC, is a specific focus on meaningful mathematical connections made by teachers during instruction, connections that are essential for fostering conceptual understanding. The Figure 11 below illustrates how our work extends a particular branch of the ETC framework.

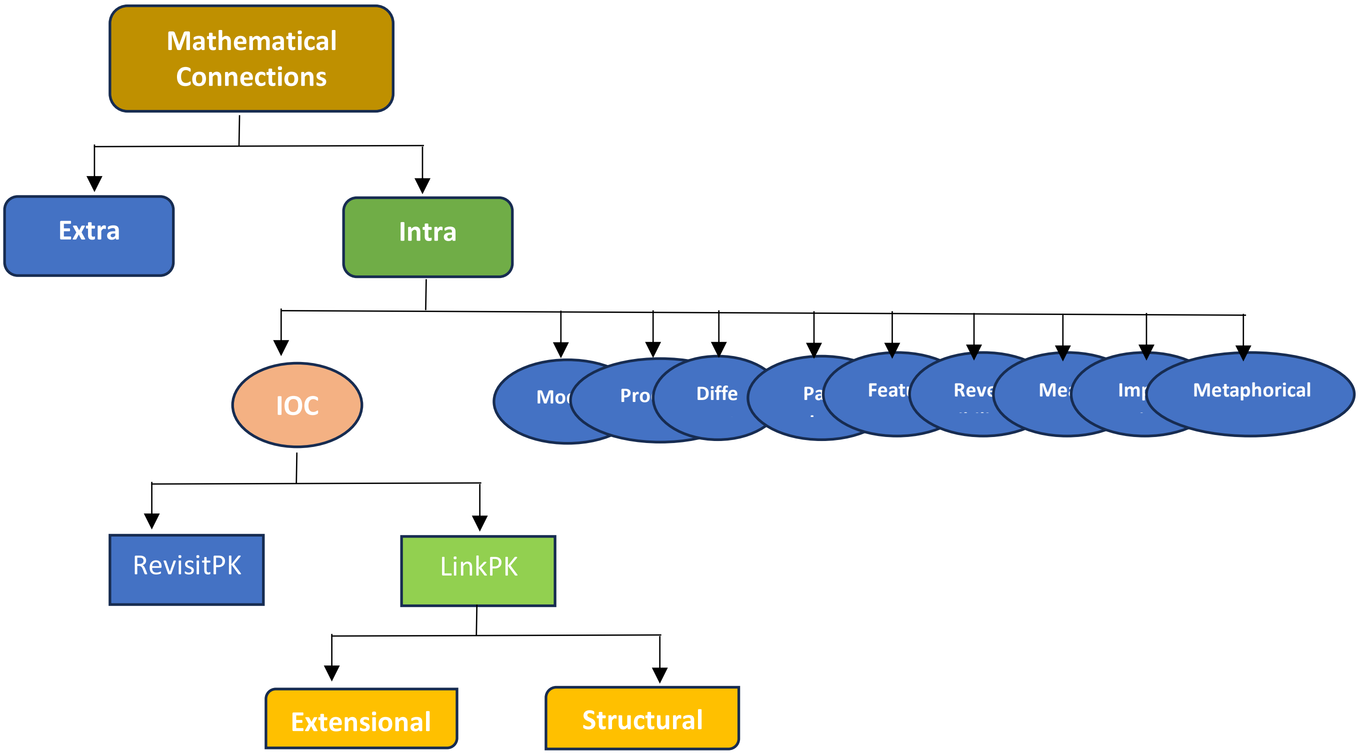


Figure 11 Placement of Extensional and Structural Connections in the ETC Framework

Recognizing and utilizing these connections in instruction can enhance student comprehension and deepen their engagement with mathematical concepts. Our findings suggest that similar connections may arise in the teaching of other mathematical topics, warranting further exploration and validation. The identification of these connection types offers practical insights for educators, providing them with strategies to strengthen students' conceptual understanding of complex numbers and beyond.

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